General Consequences of Scaling

Scaling of free energy under RG:

The partition f^n remains unchanged under RG transformation $\Rightarrow$

$$Z = \mathrm{Tr}_s \, e^{-H(\{s\}, K)} = \mathrm{Tr}_{s'} \, e^{-H(\{s'\}, K')}$$

where $\{s\}$ and $\{s'\}$ denote the original and renormalized d.o.f., and K, K' are the original and renormalized couplings. If originally, the total # of spins is N, then after RG $N' = \frac{N}{b^d}$ where d is the number of dimensions. Thus, naively one may expect

$$Z = e^{-N f(K)} \stackrel{?}{=} e^{-N' f(K')}$$

where f is the free energy density. However, the above equation is not exactly correct since one is allowed to have additional terms after RG to maintain Z constant. Consider e.g.

the 1d Ising model, where

$$Z = \mathrm{Tr}\, e^{\beta \sum_i s_i s_{i+1}} = A^{N/2}\, \mathrm{Tr}\, e^{\beta' \sum_i s'_{2i+1} s'_{2i-1}}$$

with $b=2$, $\tanh(\beta') = [\tanh(\beta)]^2$, $A = 2\sqrt{\cosh(2\beta)}$

$$\Rightarrow \quad Z = e^{-N f(\beta)} = e^{-\frac{N}{2} f(\beta') + \frac{N}{2} \log A}$$

Therefore, in general one writes

$$e^{-N f(K)} = e^{-N g(K) - N' f(K')}$$

where $g(K)$ is some constant (e.g. $\frac{1}{2}\log A$ above).

The crucial point is that f on both side of the equation is the same f^n.

Therefore, $$f(K) = g(K) - \frac{f(K')}{b^d}.$$

If we are only interested in the universal critical behavior, we may as well neglect $g(K)$ since it originates from integrating out short distance d.o.f.s., and is therefore expected to be non-singular (analytic) f^n of couplings, even at the critical point.

Denoting the singular part of the free energy as f_s, one therefore writes

$$f_s(K) = \frac{f_s(K')}{b^d}$$

Lets specialize to the Ising model. As advertised earlier, the Ising model has two relevant directions, one of them even under the Ising sym. ('temperature like direction') and the other one odd under the Ising sym. ('magnetic field like direction'). Let's denote the corresponding parameters, linearized near the RG fixed point, as:

$$u_t = t/t_0 \quad \text{where } t = \frac{T-T_c}{T_c}$$

$$u_h = h/h_0 \quad \text{where } h \text{ is the magnetic field.}$$

t_0 and h_0 are some constants.

Denoting the RG eigenvalue of u_t, u_h as y_t, y_h respectively $\Rightarrow$ renormalized $u_{t'} = b^{y_t} u_t$, $u_{h'} = b^{y_h} u_h$.

$$\Rightarrow \quad f_s(u_t, u_h) = b^{-d} f_s(u_{t'}, u_{h'})$$

$$= b^{-d} f_s(b^{y_t} u_t, b^{y_h} u_h)$$

Repeating the RG process n times, one obtains

$$f_s(u_t, u_h) = b^{-nd} f_s(b^{ny_t} u_t, b^{ny_h} u_h)$$

Let's stop the RG when $|b^{ny_t} u_t| = u_{t0}$ where u_{t0} is fixed but arbitrary.

$$\Rightarrow b^n = \left|\frac{u_{t0}}{u_t}\right|^{1/y_t}$$

$$\Rightarrow f_s(u_t, u_h) = \left|\frac{u_t}{u_{t0}}\right|^{d/y_t} f_s\left(\pm u_{t0}, u_h \left|\frac{u_t}{u_{t0}}\right|^{-y_h/y_t}\right)$$

Substituting $u_t = t/t_0$ and $u_h = h/h_0$, and keeping the dependence only on the relevant variables,

$$f_s(t,h) = \left|\frac{t}{t_0}\right|^{d/y_t} \phi\left(\frac{h/h_0}{|t/t_0|^{y_h/y_t}}\right)$$

where ϕ is a scaling f^n and is universal.

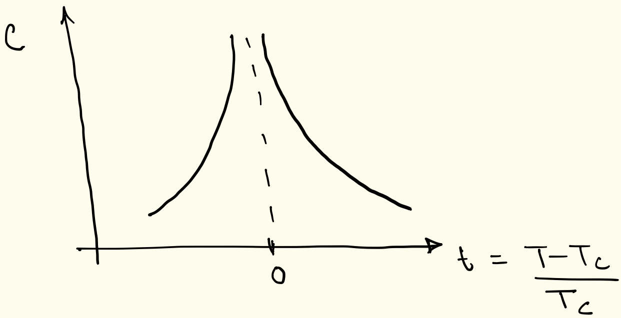
Now we are in a position to calculate critical exponents. One basic observation is that since there are only two relevant RG eigenvalues (y_t, y_h), all critical exponents can be determined solely in terms of them.

Specific heat at $h=0$, close to $t=0$:

$$C \sim \left.\frac{\partial^2 f_s}{\partial t^2}\right|_{h=0} \sim |t|^{d/y_t - 2}$$

Defining the critical exponent α as $C \sim |t|^{-\alpha}$

$$\Rightarrow \alpha = 2 - d/y_t .$$

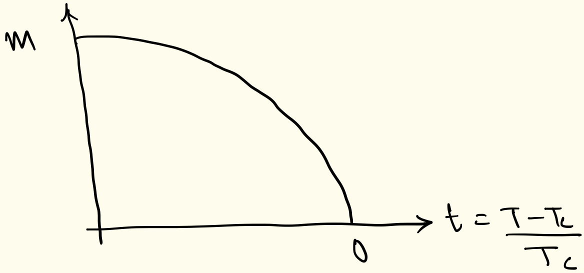


Magnetization at $h=0$, close to $t=0$:

$$m \sim \left.\frac{\partial f_s}{\partial h}\right|_{h=0} \sim |t|^{(d-y_h)/y_t}$$

Defining the critical exponent β as $m \sim (-t)^{\beta}$

$$\beta = (d - y_h)/y_t$$

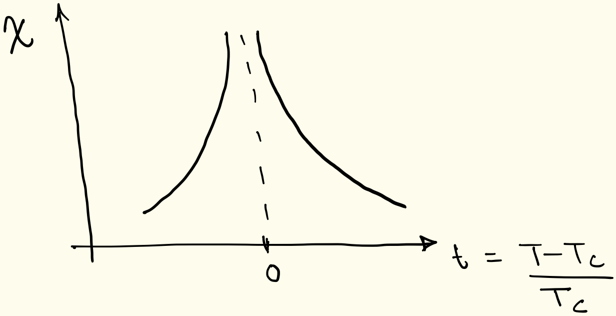


Magnetic susceptibility at $h=0$, close to $t=0$:

$$\chi \sim \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \sim |t|^{d/y_t - 2y_h/y_t}$$

Defining the exponent γ as $\chi \sim |t|^{-\gamma}$

$$\Rightarrow \quad \gamma = \frac{2y_h - d}{y_t}$$



Magnetization m as a f^n of h at $t=0$:

$$m \sim \left.\frac{\partial f}{\partial h}\right|_{t=0} \sim |t|^{d/y_t - \frac{y_h}{y_t}} \left.\phi'\left(\frac{h/h_0}{|t/t_0|^{y_h/y_t}}\right)\right|_{t=0}$$

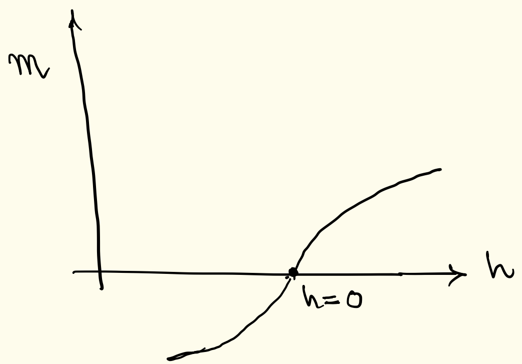
Since the above quantity must be independent of t, $\phi'(x)$ as $x \to \infty$ must scale as x^a where $\frac{d-y_h}{y_t} = \frac{a\, y_h}{y_t}$

$$\Rightarrow \quad a = (d-y_h)/y_h$$

$$\Rightarrow \quad m \sim h^a = h^{(d-y_h)/y_h}$$

Defining critical exponent δ as $m \sim h^{1/\delta}$ at $t=0$

$$\Rightarrow \quad \delta = \frac{y_h}{d-y_h}.$$



Since we defined four independent exponents using only two independent variables ($= y_h$ and y_t), there must exist two relations between the exbonents. For example, the following equations hold true as one may check:

$$\alpha + 2\beta + \gamma = 2$$
$$\alpha + \beta(1+\delta) = 2.$$

Let's verify these relations within the mean-field theory.

Landau free-energy $f = t\,m^2 + u\,m^4 - hm$

$\Rightarrow$ at $h=0$, m^* ($=$ the m that minimizes f) correspond to $m^* \sim \sqrt{\frac{-t}{u}} \Rightarrow$ exponent $\beta = \frac{1}{2}$.

Substituting $m \sim \sqrt{T_c - T}$ in $f \Rightarrow f \sim (T-T_c)^2$ $\sim t^2$ as $t \to 0$ $\Rightarrow$ $\frac{\partial^2 f}{\partial t^2} \sim$ const. $\Rightarrow \alpha = 0$

At $t=0$, minimizing f w.r.t. $m \Rightarrow m^3 u \sim h$
$\Rightarrow m \sim h^{1/3} \Rightarrow \delta = 3.$

Finally, the susceptibility $\chi = \partial m / \partial h$ satisfies

$$2r \frac{\partial m}{\partial h} + 12 u m^2 \frac{\partial m}{\partial h} = 1$$

$\Rightarrow$ for $T > T_c$, $\chi \sim \frac{1}{2r} \sim \frac{1}{T-T_c}$ $\Rightarrow$ exponent γ defined via $\chi \sim \frac{1}{(T-T_c)^\gamma}$ is $\gamma = 1$. Similarly when $T < T_c$, one finds $\gamma = 1$.

Compiling the exponents,

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3.$$

$$\alpha + 2\beta + \gamma = 0 + 1 + 1 = 2$$

$$\alpha + \beta(1+\delta) = \frac{1}{2} \times 4 = 2$$

Therefore, the above derived general relations hold, as they should.